

Ordering States with Entanglement Measures

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We demonstrate that all good asymptotic entanglement measures are either identical or place a different ordering on the set of all quantum states.

1 Introduction

The term entanglement has been used in many different ways since its inception almost fifty years ago. Although it was originally used to describe the quantum correlations of the EPR paradox and Bell's inequalities, it has recently come to be regarded as a resource that can be used to perform various quantum information processing and communication tasks [1,2]. This more application-orientated aspect of entanglement has led to important questions regarding our ability to manipulate entanglement locally. Practically, we may never have sources that generate perfect entanglement, and therefore it is necessary to explore our ability to bring (using only local operations and classical communication) diluted forms of entanglement into a more concentrated form, such as singlet states. This question has led to the development of entanglement purification methods such as those in [3–5]. Attempts to bound the efficiency of such procedures have then given rise to the concept of entanglement measures, which attempt to quantify the amount of entanglement in a given state. Indeed, a variety of entanglement measures have been proposed in recent years. Examples of various proposals can be found in [6–9].

There are two main approaches towards entanglement measures. A practical approach tackles such issues as to how much entanglement is required to generate a given state [9] or how much entanglement can be distilled from it [6], whereas a more axiomatic approach first formulates conditions any reasonable entanglement measure has to satisfy [6–8,10], and then constructs abstract measures which satisfy them [7,8]. While there is agreement on a number of basic properties, such as the non-increase of entanglement under local operations and classical communication (LQCC), it would be interesting to see what other ‘natural’ conditions one could add in order to narrow down the set of ‘acceptable’ measures. In a recent paper, M.,P. & R. Horodecki [10], for

example, considered a set of reasonable postulates in which a key ingredient was the assumption that the distillable entanglement is a well-defined measure of entanglement. Starting out from these postulates, they deduced a number of useful properties that any other entanglement measure has to satisfy.

Here we would like to ask the following question. What are the consequences if we demand that any good entanglement measure should generate the same ‘ordering’ on the set of density operators? To be precise, two entanglement measures E_1 and E_2 are said to generate the same order, if, for all density operators ρ_1 and ρ_2 we have that

$$E_1(\rho_1) \leq E_1(\rho_2) \Leftrightarrow E_2(\rho_1) \leq E_2(\rho_2) . \quad (1)$$

In this paper we show that this requirement is indeed a very strong one. In fact, we show that for most of the entanglement measures which have been proposed, this requirement can only be satisfied if the two measures are identical. A consequence of this result is that if the distillable entanglement, the entanglement of formation, and the relative entropy of entanglement are not identical, then they generate different orderings on the set of density operators. This means that we cannot conclude that the distillable entanglement of ρ_1 is smaller than that of ρ_2 just because the entanglement of formation of ρ_1 may happen to be smaller than that of ρ_2 .

While questions like this have previously been investigated numerically for special entanglement measures [11], our analytical reasoning applies to all measures that reduce to the entropy of entanglement for pure states. As it is commonly accepted that the unique measure of entanglement for pure states is the entropy of entanglement [12,13], our argument applies to all good entanglement measures.

This paper is organized as follows: in the following section we formalize the conditions under which our reasoning applies, and briefly present some important measures which satisfy them. Then we present the proof of the theorem, and finally we discuss some of the implications of our result.

2 Review of some Entanglement Measures

In order for our argument to apply, we only require the property that our entanglement measures are equal for a set of states C , such that the set of entanglement values on C , $E(\rho \in C)$, is a dense subset of the set of all entanglement values. This characteristic is true for all the entanglement measures listed below (and many more), as they are all equivalent to a particular continuous function on the set of pure states. Given a pure state $|\psi\rangle$, they all

define the entanglement as being $E(|\psi\rangle\langle\psi|) = S(\text{tr}_A(|\psi\rangle\langle\psi|))$, where $S(\rho)$ is the standard von Neumann entropy and $\text{tr}_A(\dots)$ denotes the partial trace. Indeed, as the entropy of entanglement is essentially the unique measure of entanglement for pure states [12,13], our reasoning applies to all entanglement measures. Some of the more notable candidates are briefly described as follows.

1. Entanglement of Formation, $E_F(\rho)$ [6,9]. This measure is defined as:

$$E_F(\rho) = \min_{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i E(|\psi_i\rangle\langle\psi_i|) . \quad (2)$$

Since this measure was first introduced, a slight modification has been suggested in [17]. There it was proposed that the total entanglement of formation should be

$$E_F^{\text{tot}}(\rho) = \lim_{n \rightarrow \infty} \frac{E_F(\rho^{\otimes n})}{n} , \quad (3)$$

where $\rho^{\otimes n}$ is the n -fold tensor product of ρ . The reasoning behind this is that it is not known whether $E_F(\rho)$ is additive under tensor products. It is therefore possible that the total entanglement of formation $E_F^{\text{tot}}(\rho)$ is lower than $E_F(\rho)$, and hence would be the appropriate measure of the number of singlets required to generate ρ asymptotically. The physical interpretation of the total entanglement of formation is that if Alice and Bob start out with an asymptotically large supply of singlet states, this measure determines the maximum number of copies of ρ they can generate per input singlet state using LQCC.

2. Distillable Entanglement, $E_D(\rho)$ [6,9]. If Alice and Bob start out with asymptotically many copies of ρ , this measure is defined as the maximum number of output singlet states they can obtain using LQCC per input copy of ρ .

3. Relative Entropy of Entanglement, $E_R(\rho)$ [7,8,15]. This measure is defined as

$$E_R(\rho) = \min_{\sigma \in \mathcal{D}} S(\rho || \sigma) = -S(\rho) + \min_{\sigma \in \mathcal{D}} \text{tr}(-\rho \log(\sigma)) , \quad (4)$$

where \mathcal{D} is either the set of separable states [7,8] or the set of PPT (positive partial transpose) states [16]. This measure cannot increase under LQCC and is known to be an upper bound on the distillable entanglement $E_D(\rho)$. It has proved very useful as in some cases it is an achievable upper bound to $E_D(\rho)$ [16], a function which is difficult to calculate in general. In addition, $E_R(\rho)$ is known [8,16] to lie between the entanglement of formation as defined in Eq. (2), and the distillable entanglement $E_D(\rho)$.

3 The Proposed Measures are either Equivalent or do not Possess the same Ordering

We will prove our main result for just two measures of entanglement, $E_F(\rho)$ and $E_D(\rho)$, as the proof is identical for all others. As stated earlier, these measures reduce to $S(\rho_A)$ for the pure states (in fact, all good asymptotic entanglement measures should reduce to this form on the pure states [12,13]). As this is a continuous function of the eigenvalues of ρ_A , and completely covers the range of possible values of entanglement, it is possible to find a pure state corresponding to any value of entanglement for any of the measures. Now consider a general mixed state ρ , with entanglement of formation $E_F(\rho)$. Pick two pure states $|\phi\rangle$ and $|\psi\rangle$ with entanglements of formation $E_F(|\phi\rangle) = E_F(\rho) + \epsilon$ and $E_F(|\psi\rangle) = E_F(\rho) - \epsilon$ respectively for some $\epsilon > 0$. We will start out by assuming that both $E_F(\rho)$ and $E_D(\rho)$ place the same ordering on states. As $E_F(|\phi\rangle) \geq E_F(\rho) \geq E_F(|\psi\rangle)$, this would require that

$$E_D(|\phi\rangle) \geq E_D(\rho) \geq E_D(|\psi\rangle) \quad (5)$$

but as $E_D(|\phi\rangle) = E_F(|\phi\rangle)$ and $E_D(|\psi\rangle) = E_F(|\psi\rangle)$ due to equivalence on the pure states, we would hence require that:

$$E_F(\rho) + \epsilon \geq E_D(\rho) \geq E_F(\rho) - \epsilon \quad (6)$$

Taking ϵ to zero we see that

$$E_D(\rho) = E_F(\rho) \quad (7)$$

for all states ρ , if we require both measures to place the same ordering on all states. This conclusion would be the same regardless of which measures we choose to compare, as long as they coincide on pure states.

4 Discussion and Conclusions

We have shown that we are faced with a simple choice: the equivalence and continuity of all measures on the pure states forces them to either be identical or induce a different ordering in general. This has interesting repercussions for investigations of entanglement. As most entanglement measures are very difficult to calculate in general, it might have been hoped that calculating a more tractable measure would allow us to at least work out the correct ordering for other measures. The reasoning presented here acts as a caveat against such an approach.

There are also repercussions as to whether there is a unique asymptotic measure of entanglement for mixed states. It is already known [14] that there is no single number which characterises finite transformations of pure states as effectively as $S(\rho_A)$ does in the infinite case. Although we expect that asymptotic mixed state entanglement is not characterised by a single number, this has not been proven as yet. If the total entanglement of formation E_F^{tot} given in Eq. (3) turns out to be equivalent to the distillable entanglement, then clearly one number would be sufficient. However, if this turns out not to be the case, then that would mean that $E_F^{\text{tot}}(\rho)$ would not give the same ordering as $E_D(\rho)$, which is in itself an unexpected conclusion.

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